Applied Project: Roller Derby

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May 2022

1 Executive Summary

In a purely conservational system – that is, a system where energy is conserved and is not lost to friction, drag, or heat – all energy is transformed between potential energy and kinetic energy. Therefore, if an object starts at rest at the top of a ramp and starts rolling downwards, the potential energy of the object lost from the distance that it ended up travelling vertically must equal the combined translational kinetic energy and rotational kinetic energy at each point along the slope, resulting in the following equation (for the conservation of energy equations, see Appendix A):

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{1}$$

Using this equation, and the moment of inertia equation for three-dimensional objects, the project is centered on the question of whether or not different shaped objects – a solid cylinder, a hollow cylinder, a solid sphere, and a hollow sphere – will roll to the bottom of a slope at different times, given the same slope, starting height, and starting time, and, if so, which will reach the bottom fastest and therefore win in a race.

For this project's methodology, all equations were calculated entirely by hand on whiteboards. No software was used for any calculations for this project. Using Maple to check the integrals for questions four, five, and six was attempted, but due to either software error, user error, or computational complexity, the software would freeze or crash and would not calculate the integrals given, requiring them to be solved manually.

Question one asks for a manipulation of this equation to solve for v^2. Because the object is assumed to not slide, there is a direct relationship between the angular velocity and the translational velocity, such that $\omega = v/r$. It will also be useful to know that for different objects, the coefficient of the moment of inertia I changes with the shape of the object relative to its mass and radius, in the form such that $I = I^* mr^2$. Plugging these into the above equation gives the following:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I^* mr^2 \frac{v^2}{r^2} \tag{2}$$

Through algebraic manipulation, the following equations result:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I^* mv^2 \tag{3}$$

$$gh = \frac{1}{2}v^2\left(1 + I^*\right) \tag{4}$$

$$2gh = v^2\left(1 + I^*\right) \tag{5}$$

$$v^2 = \frac{2gh}{1 + I^*} \tag{6}$$

From this form, it is already possible to see intuitively that the translational velocity of the object, given some change in height h and constant gravitational acceleration g, decreases as the coefficient of inertia I^* increases, which will be thoroughly explored throughout the rest of the project.

For question two, the above equation must be manipulated to solve for the change in the vertical distance travelled h given that the incline of the plane is some angle α. First it is important to realize that the velocity v in this equation is the scalar magnitude of the velocity vector, and does not consider direction. Because it is known that the object is rolling down an incline of slope α, if the slope α was thought of as an angle with the positive x-axis, then the vector of the velocity would be a vector with magnitude v pointing in the $a+\pi$ direction. Because the velocity is the change in position due to time, its x and y components are the change in x and y with respect to time (dx/dt and dy/dt). Therefore, the following can be used to find dy/dt:

$$\frac{dy}{dt} = v\sin(\alpha+\pi) \tag{7}$$

$$\frac{dy}{dt} = -v\sin(\alpha) \tag{8}$$

Note that because the ball is rolling down from a high height to a low height, dy/dt will be negative. However, what is asked for is not dy/dt, but rather dh/dt, the change in the total travelled vertical distance over time. Because h is a function of time such that it starts at an initial maximum height, and then the distance travelled vertically is the difference between this initial height and the y position of the object, $h(t)$ can be written as follows:

$$h(t) = y_0 - y(t) \tag{9}$$

Therefore, taking the derivative gives the following relationship between dh/dt and dy/dt:

$$\frac{dh}{dt} = -\frac{dy}{dt} \tag{10}$$

So, the change in the height travelled over time, dh/dt, is the following equation:

$$\frac{dh}{dt} = v\sin(\alpha) \tag{11}$$

Plugging in the square root of v^2 calculated earlier for v gives the following equation for dh/dt:

$$\frac{dh}{dt} = \sin(\alpha)\sqrt{\frac{2gh}{1+I^*}} \tag{12}$$

Or, splitting out the $\sqrt{h}$ from the square root:

$$\frac{dh}{dt} = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)\sqrt{h} \tag{13}$$

Question three asks to solve the differential equation above to find the total time taken to reach the bottom of the slope. By separating the equation such that the left side of the equals sign is all in terms of h, and the right side is in terms of t (and constants), the integral of both sides gives the total time taken to travel down the slope, as follows (for steps, see Appendix B):

$$t = \sqrt{\frac{2h(1+I^*)}{g\sin^2(\alpha)}} \tag{14}$$

In this equation, between the four objects that are being rolled down the slope, the change in height h, the acceleration due to gravity g, and the angle of the slope α are all exactly the same (i.e. constant), and therefore it is clear to see that the only variable which changes the time it takes for the objects to roll down the slope is the coefficient of the moment of inertia, I^*. The greater the coefficient, the slower it rolls down the slope and, therefore, the longer it takes to reach the bottom. Going back to the original conservation of energy equation, this is because at each point that the object is rolling down the slope, the energy that was converted from gravitation potential energy to kinetic energy has more stored in rotational kinetic energy and not the translational kinetic energy for objects with a higher coefficient of inertia. The translational kinetic energy, and therefore the translational velocity, is ultimately what effects the object's time to reach the bottom of the slope, and therefore, the lower the coefficient of inertia, the faster the object will reach the bottom of the slope to win the race.

The following three questions ask for the mathematical values of the coefficients of inertia for objects of four shapes: a solid cylinder, a hollow cylinder, a solid sphere, and a hollow sphere. Question four starts by asking for the coefficients of solid and hollow cylinders. These can be found using the moment of inertia triple integral equation, which states that the moment of inertia of an object with density $\rho(x,y,z)$ about some axis, such as the z-axis, is given by the following:

$$I_z = \iiint_V \left[\left(x^2+y^2\right) * \rho(x,y,z)\right] dV \tag{15}$$

For all four of the objects, it will be assumed that the density is a constant, such that $\rho(x,y,z) = k$. Thus, all that must be found to complete the integral is an expression for the volume of a cylinder as a triple integral. This is most readily accomplished using the cylindrical coordinate system, where a solid cylinder can be described as $0 \le r \le R$, where R is the cylinder's radius as a constant, and a hollow cylinder can be described with $A \le r \le R$, with A being the internal radius and R being the external radius. The height of the cylinder can be described with $0 \le z \le L$, and θ will be a full rotation such that $0 \le \theta < 2\pi$. The x^2+y^2 in the integrand can also be converted to cylindrical using the equation $x^2+y^2 = r^2$. By using the second version for the hollow cylinder as a general case (whereby a solid cylinder can also be found by simply setting $A = 0$), the following triple integral gives the moment of inertia about the z-axis:

$$I_z = \int_0^{2\pi}\int_0^{L}\int_A^{R} kr^3\, dr\, dz\, d\theta \tag{16}$$

This ultimately gives the following result (for integration, see Appendix C):

$$I_z = \frac{1}{2}\pi k L \left(R^4 - A^4\right) \tag{17}$$

Looking at the shape of a solid cylinder, it is known that the mass of a solid cylinder with density k is simply the volume times the density, however it is important to consider that the volume of a hollow cylinder, or a generalized equation for either a hollow or solid cylinder, is not simply $\pi R^2 L$, because there's an inside and outside radius to this equation in the hollow or general case. Therefore, the mass m is given by the following equation:

$$m = \pi k R^2 L - \pi k A^2 L \tag{18}$$

$$m = \pi k L \left(R^2 - A^2\right) \tag{19}$$

To find the coefficient of the moment of inertia, recall that $I^* = I/mr^2$. Substituting our moment of inertia and our mass into this equation results in the following:

$$I^* = \frac{\frac{1}{2}\pi k L \left(R^4 - A^4\right)}{\pi k L \left(R^2 - A^2\right) R^2} \tag{20}$$

In the case that $A = 0$ (a solid cylinder), the following equation is given:

$$I^* = \frac{\frac{1}{2}\pi k L R^4}{\pi k L R^4} \tag{21}$$

$$I^* = \frac{1}{2} \tag{22}$$

Therefore, a solid cylinder has a coefficient of $1/2$. For a hollow cylinder, the limit of this equation must be considered as $A \to R$ to see the behavior as the cylinder gets more and more hollow, which gives the following:

$$I^* = \lim_{A \to R} \frac{\frac{1}{2}\pi k L \left(R^4 - A^4\right)}{\pi k L \left(R^2 - A^2\right) R^2} = \frac{0}{0} \tag{23}$$

Through algebraic simplification:

$$I^* = \lim_{A \to R} \frac{\frac{1}{2}\pi k L \left(R^2 - A^2\right)\left(R^2 + A^2\right)}{\pi k L \left(R^2 - A^2\right) R^2} \tag{24}$$

$$= \lim_{A \to R} \frac{\frac{1}{2}\left(R^2 + A^2\right)}{R^2} \tag{25}$$

$$= \frac{\frac{1}{2}\left(2R^2\right)}{R^2} = 1 \tag{26}$$

Therefore, the coefficient for a perfectly hollow cylinder is 1. An alternate way to think about the moment of inertia for a perfectly hollow cylinder is to remember that the moment of inertia is the sum of infinitesimal pieces of mass times their radius from the axis of rotation squared. Because a hypothetically infinitely thin cylinder has all of its mass exactly the radius of the cylinder away from the axis of rotation, the moment of inertia would be such that $I = mr^2$, and the coefficient is therefore 1.

Question five asks for the same process to be repeated but for a sphere. Similar to how the cylinder was converted to cylindrical coordinates, the sphere will also be converted to spherical coordinates for the moment of inertia triple integral, where the sphere is described such that $A \leq \rho \leq R$ (where A and R are the internal and external radii), $0 \leq \phi \leq \pi$, and $0 \leq \theta < 2\pi$. Choosing to rotate around the z-axis once again, $x^2 + y^2 = r^2 = \rho^2 \sin^2(\phi)$, giving the following integral:

$$I_z = \int_0^{2\pi} \int_0^{\pi} \int_A^{R} k\rho^4 \sin^3(\phi)\, d\rho\, d\phi\, d\theta \tag{27}$$

Calculating the integral (using basic techniques and only one u-substitution, see Appendix D) gives the following result:

$$I_z = \frac{2}{5}\left(\frac{4}{3}\right)\pi k\left(R^5 - A^5\right) \tag{28}$$

The mass of a partially hollow sphere can be given by the volume times the density k, which gives the following equation:

$$m = \frac{4}{3}\pi k\left(R^3 - A^3\right) \tag{29}$$

By letting $B = A/R$, the equations can be rewritten as follows by factoring out powers of R:

$$I_z = \frac{2}{5}\left(\frac{4}{3}\right)\pi k R^5\left(1 - B^5\right) \tag{30}$$

$$m = \frac{4}{3}\pi k R^3\left(1 - B^3\right) \tag{31}$$

Therefore, I^* can be expressed thus:

$$I^* = \frac{\frac{2}{5}\left(\frac{4}{3}\right)\pi k R^5\left(1 - B^5\right)}{\frac{4}{3}\pi k R^3\left(1 - B^3\right)R^2} \tag{32}$$

$$I^* = \frac{2}{5}\left(\frac{1-B^5}{1-B^3}\right) \tag{33}$$

As $A \to R$, $B \to 1$, and as $A \to 0$, $B \to 0$. Therefore, to answer question six, a solid sphere would be such that $B = 0$, and a hollow sphere would be the limit as $B \to 1$. Plugging in 0 for B, the coefficient of inertia becomes $2/5$. Then, the limit as $B \to 1$ gives the following:

$$I^* = \lim_{B \to 1} \frac{2}{5}\left(\frac{1-B^5}{1-B^3}\right) = \frac{0}{0} \tag{34}$$

By L'Hôpital's rule:

$$I^* = \lim_{B \to 1} \frac{2}{5}\left(\frac{-5B^4}{-3B^2}\right) = \frac{2}{5}\left(\frac{5}{3}\right) = \frac{2}{3} \tag{35}$$

Therefore, the coefficient of inertia of a hollow sphere approaches $2/3$.

To conclude, because it was determined in question three that the object with the lowest coefficient of inertia would reach the bottom of the slope first, the objects can now be listed in the order in which they complete the race:

1. Solid sphere, $I^* = 2/5$
2. Solid cylinder, $I^* = 1/2$
3. Hollow sphere, $I^* = 2/3$
4. Hollow cylinder, $I^* = 1$

2 Appendices

2.1 Appendix A

In a conservational system, where energy is not lost to heat or friction and all of the kinetic and potential energy from the beginning of observation totals to the same amount of Joules of kinetic and potential energy at the end of observation, the system can be described with the following equation:

$$KE_i + PE_i = KE_f + PE_f \tag{36}$$

For this project, four objects with circular cross-sections are considered, and the problem posed is to manipulate the above equation in such a way as to determine which of the four objects will roll to the bottom of the ramp first starting at rest given their different shapes (a solid cylinder, a hollow cylinder, a solid sphere, and a hollow sphere).

Because the objects all start at rest, their initial kinetic energy will be zero. Similarly, because potential energy is relative to the change in height, the final potential energy can be ignored. This leaves the following relevant equation:

$$PE_i = KE_f \tag{37}$$

The potential energy of each object is simply the mass times the acceleration due to gravity and the height, or $PE_i = mgh$ (the h can be thought of as a change in height, or Δy, thereby accounting for the difference between the energy at the starting height and the energy at the ending height). Kinetic energy of a rotating body, likewise, is made up of rotational kinetic energy and translational kinetic energy, such that:

$$KE_f = KE_t + KE_r \tag{38}$$

The translational kinetic energy is given by $KE_t = (1/2)\,mv^2$, where m is the mass of the object and v is the translational velocity, and the rotational kinetic energy is $KE_r = (1/2)I\omega^2$, where I is the moment of inertia of the object and ω is the angular velocity. Therefore, the following conservation of energy equation results:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{39}$$

2.2 Appendix B

$$\frac{1}{\sqrt{h}}\,dh = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)\,dt \tag{40}$$

$$\int h^{-\frac{1}{2}}\,dh = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)\int dt \tag{41}$$

$$2\sqrt{h} = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)\,t \tag{42}$$

$$t = \frac{2\sqrt{h}}{\sqrt{\frac{2g}{1+I^*}}\sin(\alpha)} \tag{43}$$

$$t = \frac{\sqrt{4h}}{\sqrt{\frac{2g\sin^2(\alpha)}{1+I^*}}} \tag{44}$$

$$t = \sqrt{\frac{4h(1+I^*)}{2g\sin^2(\alpha)}} \tag{45}$$

$$t = \sqrt{\frac{2h(1+I^*)}{g\sin^2(\alpha)}} \tag{46}$$

2.3 Appendix C

$$I_z = k\int_0^{2\pi}\int_0^{L}\int_A^{R} r^3\,dr\,dz\,d\theta \tag{47}$$

$$= k\int_0^{2\pi}\int_0^{L}\left(\frac{R^4}{4}-\frac{A^4}{4}\right)dz\,d\theta \tag{48}$$

$$= \frac{k}{4}\left(R^4-A^4\right)\int_0^{2\pi}\int_0^{L} dz\,d\theta \tag{49}$$

$$I_z = \frac{k}{4}\left(R^4-A^4\right)2\pi L \tag{50}$$

2.4 Appendix D

$$I_z = k\int_0^{2\pi}\int_0^{\pi}\int_A^{R} \rho^4\sin^3(\phi)\,d\rho\,d\phi\,d\theta \tag{51}$$

$$= k\left(\frac{R^5}{5}-\frac{A^5}{5}\right)\int_0^{2\pi}\int_0^{\pi}\sin^3(\phi)\,d\phi\,d\theta \tag{52}$$

$$= k\left(\frac{R^5}{5}-\frac{A^5}{5}\right)\int_0^{2\pi}\int_0^{\pi}\sin(\phi)\left(1-\cos^2(\phi)\right)d\phi\,d\theta \tag{53}$$

$$= k\left(\frac{R^5}{5}-\frac{A^5}{5}\right)\int_0^{2\pi}\int_1^{-1}\left(u^2-1\right)du\,d\theta \tag{54}$$

$$= k\left(\frac{R^5}{5}-\frac{A^5}{5}\right)\left(\frac{4}{3}\right)\int_0^{2\pi}d\theta \tag{55}$$

$$= k\left(\frac{R^5}{5}-\frac{A^5}{5}\right)\left(\frac{4}{3}\right)2\pi \tag{56}$$

$$I_z = \frac{2}{5}\left(\frac{4}{3}\right)\pi k\left(R^5-A^5\right) \tag{57}$$

3 License and information

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